Quantum three-body system in D dimensions

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The independent eigenstates of the total orbital angular momentum operators for a three-body system in an arbitrary D-dimensional space are presented by the method of group theory. The Schrödinger equation is reduced to the generalized radial equations satisfied by the generalized radial functions with a given total orbital angular momentum denoted by a Young diagram $[\mu, \nu, 0, \ldots, 0]$ for the SO(D) group. Only three internal variables are involved in the functions and equations. The number of both the functions and the equations for the given angular momentum is finite and equal to $(\mu - \nu + 1)$.

I. INTRODUCTION

From the very early stage of the progress in quantum mechanics in the real three-dimensional world, it has been pointed out that the essence of these theories would be easily understandable if their mathematics is constructed in the non-relativistic hyper-space worlds [1,2]. The mathematical tools for generalization of the orbital angular momentum in an arbitrary D-dimensional space have been presented [3–7]. Recently, the D-dimensional Coulombic and the harmonic oscillator problems in a two-body system have been studied in some detail by many authors [8–22].

Exact solutions played very important roles in the development of physics. The exact solutions of the Schrödinger equation in the real three dimensional space for a hydrogen atom and for a harmonic oscillator were important technical achievements in quantum

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mechanics [23], which provided strong evidence in favor of the theory being correct, at least as far as atomic physics is concerned. The next simplest atom is the helium atom, for which the Schrödinger equation cannot be solved analytically, but only numerically [24–28]. In the numerical calculation, one of the main difficulties is how to separate the global rotational degrees of freedom.

In our previous paper [29] we separated completely the global rotational degrees of freedom in the Schrödinger equation for an N-body system in the real three-dimensional space from the internal ones. We have determined a complete set of $(2l+1)$ independent base functions for a given total orbital angular momentum l, which are the homogeneous polynomials in the components of coordinate vectors and do not contain the Euler angles explicitly. Any function with the given angular momentum l in the system can be expanded with respect to the base functions, where the coefficients are the functions of the internal variables, called the generalized radial functions. The generalized radial equations satisfied by the functions are established explicitly [29]. For the typical three-body system in the real three dimensional space [30,31], such as a helium atom [32,33] and a positronium negative ion [34], the generalized radial equations [35] have been solved numerically with high precision.

With the interest of higher dimensional field theory recently, we have a try to generalize the study of the D-dimensional two-body system to the D-dimensional three-body system. The purpose of this paper is, for a three-body system in an arbitrary D-dimensional space, to find a complete set of independent base functions with any given total orbital angular momentum and to reduce the Schrödinger equation with a spherically symmetric potential V to the generalized radial equations, where only three internal variables are involved. Any function with the given angular momentum in the system can be expanded with respect to the base functions. It provides a possibility to calculate numerically the energy levels of the three-body system in D-dimensions with high precision.

From the viewpoint of mathematics, the separation of the global rotational degrees of freedom from the internal ones is a typical application of group theory to physics. The properties of the independent base functions for a given total orbital angular momentum would be more clear if they are constructed in arbitrary D-dimensional space than that in the real three dimensional space. The total orbital angular momentum for a three-body

system in a D-dimensional space is described by an irreducible representation denoted by a Young diagram with one or two rows. For the real three-dimensional space, the rotational symmetry group is SO(3) group, and its only irreducible representations denoted by the Young diagrams with two rows are $[l,1]$, which are equivalent to the representations denoted by the one-row Young diagrams $[l,0]$, respectively. This is the reason why the angular momentum can be described by only one quantum number l for the real three-dimensional space.

This paper is organized as follows. After separating the motion of the center of mass by the Jacobi coordinate vectors in Sec. 2, we review in Sec. 3 the generalization of the orbital angular momentum operators and the properties of the spherical harmonics [4,6] and the harmonic polynomials [7] for a two-body system in D dimensions. In Sec. 4 we will define the generalized harmonic polynomials for a three-body system in D dimensions and prove that they constitute a complete set of independent base functions for a given total orbital angular momentum in the system. The generalized radial functions are defined and the generalized radial equations are derived in Sec. 5. Some conclusions will be given in Sec. 6.

II. SCHRÖDINGER EQUATION IN D DIMENSIONS

For a quantum N-body system in an arbitrary D-dimensional space, we denote the position vectors and the masses of N particles by $\mathbf{r}_k$ and by m_k, $k = 1, 2, \ldots, N$, respectively. $M = \sum_k m_k$ is the total mass. The Schrödinger equation for the N-body system with a pair potential V, depending upon the distance of each pair of particles, $|\mathbf{r}_j - \mathbf{r}_k|$, is

$$-\frac{1}{2}\sum_{k=1}^{N} m_k^{-1} \bigtriangledown_{\mathbf{r}_k}^2 \Psi + V\Psi = E\Psi, \tag{1}$$

where $\bigtriangledown_{\mathbf{r}_k}^2$ is the Laplace operator with respect to the position vector $\mathbf{r}_k$. For simplicity, the natural units $\hbar = c = 1$ are employed throughout this paper. The total orbital angular momentum operators L_{ab} in D dimensions are defined as [4,6]

$$L_{ab} = -L_{ba} = -i\sum_{k=1}^{N} \left\{ r_{ka}\frac{\partial}{\partial r_{kb}} - r_{kb}\frac{\partial}{\partial r_{ka}} \right\}, \qquad a, b = 1, 2, \ldots D, \tag{2}$$

where r_{ka} denotes the ath component of the position vector $\mathbf{r}_k$.

Now, we replace the position vectors $\mathbf{r}_k$ by the Jacobi coordinate vectors $\mathbf{R}_j$:

$$\mathbf{R}_0 = M^{-1/2} \sum_{k=1}^{N} m_k \mathbf{r}_k, \qquad \mathbf{R}_j = \left(\frac{m_{j+1} M_j}{M_{j+1}} \right)^{1/2} \left(\mathbf{r}_{j+1} - \sum_{k=1}^{j} \frac{m_k \mathbf{r}_k}{M_j} \right),$$

$$1 \leq j \leq (N-1), \qquad M_j = \sum_{k=1}^{j} m_k, \qquad M_N = M, \tag{3}$$

where $\mathbf{R}_0$ describes the position of the center of mass, $\mathbf{R}_1$ describes the mass-weighted separation from the second particle to the first particle. $\mathbf{R}_2$ describes the mass-weighted separation from the third particle to the center of mass of the first two particles, and so on. An additional factor $\sqrt{M}$ is included in $\mathbf{R}_j$ for convenience. The mass-weighted factors in front of the formulas for $\mathbf{R}_j$ are determined by the condition

$$\sum_{k=1}^{N} m_k \mathbf{r}_k^2 = \sum_{j=0}^{N-1} \mathbf{R}_j^2,$$

One may determine the factors one by one from the following schemes. In the center-of-mass frame, if the first j particles coincide with each other and the last $(N-j-1)$ particles are located at the origin, the factor in front of $\mathbf{R}_j$ is determined by

$$\mathbf{r}_1 = \mathbf{r}_2 = \cdots = \mathbf{r}_j = -m_{j+1} \mathbf{r}_{j+1} / M_j, \qquad \sum_{k=1}^{j+1} m_k \mathbf{r}_k^2 = \mathbf{R}_j^2. \tag{4}$$

A straightforward calculation by replacement of variables shows that the Laplace operator in Eq. (1) and the total orbital angular momentum operator L_{ab} in Eq. (2) are directly expressed in $\mathbf{R}_j$:

$$\begin{aligned} \nabla^2 &= \sum_{k=1}^{N} m_k^{-1} \nabla_{\mathbf{r}_k}^2 = \sum_{j=0}^{N-1} \nabla_{\mathbf{R}_j}^2, \\ L_{ab} &= -i \sum_{j=0}^{N-1} \left\{ R_{ja} \frac{\partial}{\partial R_{jb}} - R_{jb} \frac{\partial}{\partial R_{ja}} \right\}. \end{aligned} \tag{5}$$

In the center-of-mass frame, $\mathbf{R}_0 = 0$. The Laplace operator (5) obviously has the symmetry of the $\mathrm{O}(ND-D)$ group with respect to $(N-1)D$ components of $(N-1)$ Jacobi coordinate vectors. The $\mathrm{O}(ND-D)$ group contains a subgroup $\mathrm{SO}(D) \times \mathrm{O}(N-1)$, where $\mathrm{SO}(D)$ is the rotation group in the D-dimensional space. The space inversion and the different definitions for the Jacobi coordinate vectors in the so-called Jacobi tree [24] can be obtained by $\mathrm{O}(N-1)$ transformations. For the system of identical particles, the permutation group among particles is also a subgroup of the $\mathrm{O}(N-1)$ group [29].

It is easy to obtain the inverse transformation of Eq. (3):

$$\mathbf{r}_j = \left[\frac{M_{j-1}}{m_j M_j}\right]^{1/2} \mathbf{R}_{j-1} - \sum_{k=j}^{N-1} \left[\frac{m_{k+1}}{M_k M_{k+1}}\right]^{1/2} \mathbf{R}_k + M^{-1/2}\mathbf{R}_0,$$

$$\mathbf{r}_j - \mathbf{r}_k = \left[\frac{M_j}{m_j M_{j-1}}\right]^{1/2} \mathbf{R}_{j-1} + \sum_{i=k}^{j-2} \left[\frac{m_{i+1}}{M_i M_{i+1}}\right]^{1/2} \mathbf{R}_i - \left[\frac{M_{k-1}}{m_k M_k}\right]^{1/2} \mathbf{R}_{k-1}. \tag{6}$$

Thus, the potential V is a function of $\mathbf{R}_j \cdot \mathbf{R}_k$ and is rotationally invariant.

III. HARMONIC POLYNOMIALS IN D DIMENSIONS

In the center-of-mass frame, $\mathbf{R}_0 = 0$. Hence, for a two-body system there is only one Jacobi coordinate vector $\mathbf{R}_1$, which will be denoted by $\mathbf{x}$ for simplicity:

$$\begin{aligned} \mathbf{x} &= \left(\frac{m_1 m_2}{m_1 + m_2}\right)^{1/2} \{\mathbf{r}_2 - \mathbf{r}_1\}, \\ \bigtriangledown^2 = \bigtriangledown^2_{\mathbf{x}}, \qquad L_{ab} &= -i\left\{x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a}\right\}, \end{aligned} \tag{7}$$

Louck [4,6] introduced the hyperspherical coordinates

$$\begin{aligned} x_1 &= r\cos\theta_1 \sin\theta_2 \ldots \sin\theta_{D-1}, \\ x_2 &= r\sin\theta_1 \sin\theta_2 \ldots \sin\theta_{D-1}, \\ x_k &= r\cos\theta_{k-1} \sin\theta_k \ldots \sin\theta_{D-1}, \qquad 3 \le k \le D-1, \\ x_D &= r\cos\theta_{D-1}. \end{aligned} \tag{8}$$

The spherical harmonics $Y^l_{l_{D-2},\cdots,l_1}$ in D dimensions [4,6] are the simultaneous eigenfunctions of the commutant operators $\mathbf{L}^2_k$:

$$\mathbf{L}^2_1 = -\frac{\partial^2}{\partial\theta_1^2}, \qquad \mathbf{L}^2_k = -\left\{\frac{1}{\sin^{k-1}\theta_k}\frac{\partial}{\partial\theta_k}\sin^{k-1}\theta_k \frac{\partial}{\partial\theta_k} - \frac{\mathbf{L}^2_{k-1}}{\sin^2\theta_k}\right\}, \tag{9}$$

$$\begin{aligned} &\mathbf{L}^2_1 Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1}) = l_1^2 Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1}), \\ &\mathbf{L}^2_k Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1}) = l_k(l_k+k-1) Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1}), \\ &l \equiv l_{D-1} = 0, 1, \ldots, \qquad l_k = 0, 1, \ldots, l_{k+1}, \qquad l_1 = -l_2, -l_2+1, \ldots, l_2-1, l_2, \end{aligned} \tag{10}$$

where $\mathbf{L}^2 \equiv \mathbf{L}^2_{D-1}$, $0 \le r < \infty$, $-\pi \le \theta_1 \le \pi$, $0 \le \theta_k \le \pi$, and $2 \le k \le D-1$. The volume element of the configuration space is [4,14]

$$\prod_{j=1}^{D} dx_j = r^{D-1} dr \prod_{j=1}^{D-1} (\sin\theta_j)^{j-1}\, d\theta_j. \tag{11}$$

Through a direct calculation by replacement of variables, one obtains [4,6]

$$\nabla^2_{\mathbf{x}} = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}r^{D-1}\frac{\partial}{\partial r} - \frac{\mathbf{L}^2}{r^2}, \tag{12}$$

Due to the spherical symmetry, the wave function can be expressed as

$$\psi^l_{l_{D-2},\cdots,l_1}(\mathbf{x}) = \phi_l(r)Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1}), \tag{13}$$

and the D-dimensional Schrödinger equation (1) for a two-body system in the center-of-mass frame reduces to the radial equation

$$\frac{1}{r^{D-1}}\frac{\partial}{\partial r}r^{D-1}\frac{\partial}{\partial r}\phi_l(r) - \frac{l(l+D-2)}{r^2}\phi_l(r) = -2\left[E - V(r)\right]\phi_l(r). \tag{14}$$

Bander and Itzykson [7] introduced the harmonic polynomials in D dimensions

$$\mathcal{Y}^l_{l_{D-2},\cdots,l_1}(\mathbf{x}) = r^l Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1}) \equiv r^l Y^l_{l_{D-2},\cdots,l_1}(\hat{\mathbf{x}}), \tag{15}$$

to avoid the angular functions $Y^l_{l_{D-2},\cdots,l_1}(\theta_1 \ldots \theta_{D-1})$. $\mathcal{Y}^l_{l_{D-2},\cdots,l_1}(\mathbf{x})$ is a homogeneous polynomial of degree l in the components of $\mathbf{x}$ and satisfies the Laplace equation

$$\nabla^2_{\mathbf{x}}\mathcal{Y}^l_{l_{D-2},\cdots,l_1}(\mathbf{x}) = 0. \tag{16}$$

The number of linearly independent homogeneous polynomials of degree l in D components of $\mathbf{x}$ is $N(l) = (l+D-1)!/l!(D-1)!$. The Laplace equation (16) gives $N(l-2) = (l+D-3)!/(l-2)!(D-1)!$ constraints. Hence, the number of the harmonic polynomials $\mathcal{Y}^l_{l_{D-2},\cdots,l_1}(\mathbf{x})$ of degree l as well as the number of the spherical harmonics $Y^l_{l_{D-2},\cdots,l_1}(\hat{\mathbf{x}})$ in D dimensions is

$$N(l) - N(l-2) = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!} = d_D([l,0,\ldots,0]). \tag{17}$$

$d_D([l,0,\ldots,0])$ is the dimension of the irreducible representation of SO(D) denoted by the one-row Young diagram $[l,0,\ldots,0]$. $[l,0,\ldots,0]$ describes the symmetric traceless tensor representation. In fact, any polynomial in the components of one vector $\mathbf{x}$ has to belong to a symmetric representation.

Due to the spherical symmetry, one only needs to write the explicit form of the highest weight state [7]

$$\mathcal{Y}^l_{l,\cdots,l}(\mathbf{x}) = N_l(x_1 + ix_2)^l. \tag{18}$$

where N_l denotes the normalization factor. The partners of $\mathcal{Y}^l_{l,\cdots,l}(\mathbf{x})$ can be simply generated by rotation. Now, the solution to the Schrödinger equation in the center-of-mass frame can be re-expressed as

$$\psi^l_{l,\cdots,l}(\mathbf{x}) = R_l(r)\mathcal{Y}^l_{l,\cdots,l}(\mathbf{x}), \tag{19}$$

and the radial equation is easy to be derived:

$$\frac{1}{r^{D-1}}\frac{\partial}{\partial r}r^{D-1}\frac{\partial}{\partial r}R_l(r) + \frac{2l}{r}\frac{\partial}{\partial r}R_l(r) = -2\left[E - V(r)\right]R_l(r). \tag{20}$$

Recall $R_l(r) = r^{-l}\phi_l(r)$. Eq. (20) coincides with Eq. (14) but the angle variables do not appear explicitly in calculation.

The number (17) of the harmonic polynomials $\mathcal{Y}^l_{l_{D-2},\cdots,l_1}(\mathbf{x})$ of degree l can be understood from another viewpoint. After removing those homogeneous polynomials in the form $r^2 f(\mathbf{x})$, where $f(\mathbf{x})$ is a homogeneous polynomial of degree $(l-2)$, Eq. (17) shows the number of the remaining linearly independent homogeneous polynomials of degree l in the components of $\mathbf{x}$. Therefore, the harmonic polynomials $\mathcal{Y}^l_{l_{D-2},\cdots,l_1}(\mathbf{x})$ construct a complete set of linearly independent base functions for the homogeneous polynomials of degree l in the components of $\mathbf{x}$, excluded those in the form of $r^2 f(\mathbf{x})$.

IV. THREE-BODY SYSTEM IN D-DIMENSIONS

For a three-body system, in the center-of-mass frame there are two Jacobi coordinate vectors $\mathbf{R}_1$ and $\mathbf{R}_2$, which will be denoted by $\mathbf{x}$ and $\mathbf{y}$, respectively:

$$\mathbf{x} = \left[\frac{m_1 m_2}{m_1+m_2}\right]^{1/2}\left\{\mathbf{r}_2 - \mathbf{r}_1\right\}, \qquad \mathbf{y} = \left[\frac{(m_1+m_2)m_3}{m_1+m_2+m_3}\right]^{1/2}\left\{\mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}\right\},$$

$$\begin{aligned} \bigtriangledown^2 &= \bigtriangledown^2_{\mathbf{x}} + \bigtriangledown^2_{\mathbf{y}}, \\ L_{ab} &= L^{(x)}_{ab} + L^{(y)}_{ab} = -i\left\{x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a}\right\} - i\left\{y_a\frac{\partial}{\partial y_b} - y_b\frac{\partial}{\partial y_a}\right\}, \end{aligned} \tag{21}$$

The Schrödinger equation (1) reduces to

$$\begin{aligned} \left\{\bigtriangledown^2_{\mathbf{x}} + \bigtriangledown^2_{\mathbf{y}}\right\}\Psi(\mathbf{x},\mathbf{y}) &= -2\left\{E - V(\xi_1,\xi_2,\xi_3)\right\}\Psi(\mathbf{x},\mathbf{y}), \\ \xi_1 = \mathbf{x}\cdot\mathbf{x}, \qquad \xi_2 = \mathbf{y}\cdot\mathbf{y}, &\qquad \xi_3 = \mathbf{x}\cdot\mathbf{y}. \end{aligned} \tag{22}$$

where ξ_j are the internal variables. Since Eq. (22) is rotational invariant, the total orbital angular momentum is conserved. The wave function $\Psi(\mathbf{x},\mathbf{y})$ with the given total angular momentum has to belong to an irreducible representation of SO(D). In the traditional method, one calculates the wave function by the Clebsch-Gordan coefficients:

$$\sum_{l_{D-2},\cdots,l_1 l'_{D-2},\cdots,l'_1} \mathcal{Y}^{l}_{l_{D-2},\cdots,l_1}(\mathbf{x})\mathcal{Y}^{l'}_{l'_{D-2},\cdots,l'_1}(\mathbf{y})\langle l, l_{D-2},\ldots,l_1; l', l'_{D-2},\ldots,l_1|L,M\rangle. \tag{23}$$

As usual, $\mathcal{Y}^{l}_{l_{D-2},\cdots,l_1}(\mathbf{x})$ and $\mathcal{Y}^{l'}_{l'_{D-2},\cdots,l'_1}(\mathbf{y})$ are called the partial angular momentum states, and their combination is called the total angular momentum state, which is a homogeneous polynomial of degrees l and l' in the components of $\mathbf{x}$ and $\mathbf{y}$, respectively.

There are three problems. First, what kinds of representations (or total angular momentum L) appear in the Clebsch-Gordan series for decomposition of the direct product of two representations denoted by one-row Young diagrams $[l,0,\ldots,0]$ and $[l',0,\ldots,0]$? This problem has been solved in group theory by the Littlewood-Richardson rule and traceless conditions. A new character is that the representations denoted by two-row Young diagrams appear in the Clebsch-Gordan series for a three-body system when $D>3$. Those representations denoted by the Young diagrams with more than two rows could not appear because there are only two Jacobi coordinate vectors. For simplicity we denote a one-row or two-row Young diagram by $[\mu,\nu]\equiv[\mu,\nu,0,\ldots,0]$. Hence, we have the Clebsch-Gordan series:

$$[l,0]\otimes[l',0]\simeq\bigoplus_{s=0}^{n}\bigoplus_{t=0}^{n-s}[l+l'-s-2t,s], \tag{24}$$

where n is the minimum between l and l'. The representations with $t=0$ are calculated by the Littlewood-Richardson rule, and the remaining are calculated by the traceless conditions. The dimension of a representation denoted by a two-row Young diagram is

$$\begin{aligned} d_D([\mu,\nu]) &= (D+2\mu-2)(D+\mu+\nu-3)(D+2\nu-4)(\mu-\nu+1) \\ &\quad\times\frac{(D+\mu-4)!(D+\nu-5)!}{(\mu+1)!\nu!(D-2)!(D-4)!}. \end{aligned} \tag{25}$$

When $D=4$, the representation denoted by a two-row Young diagram reduces to a direct sum of a selfdual representation $[(S)\mu,\nu]$ and an antiselfdual one $[(A)\mu,\nu]$. Their dimensions are equal to half of $d_4([\mu,\nu])$ given in Eq. (25). When $D=3$, due to the traceless condition, the only representations with the two-row Young diagrams are representations $[\mu,1]$, which are equivalent to that with the one-row Young diagrams $[\mu,0]$,

respectively. Eq. (25) still holds for $D = 3$. Second, how to calculate the Clebsch-Gordan coefficients? The calculation must be very complicated. We will avoid the difficulty by the method of determining the highest weight states directly. Third, how many base functions are independent for a given total orbital angular momentum such that any wave function with the same angular momentum can be expanded with respect to the base functions where the coefficients are the functions of the internal variables. We are going to solve the last two problems by group theory.

Let us sketch some necessary knowledge of group theory. From the representation theory of Lie groups [36–38], the Lie algebras of the SO($2n$+1) group and the SO($2n$) group are B_n and D_n, respectively. Their Chevalley bases with the subscript j, $1 \leq j \leq n-1$, are same:

$$\begin{aligned} H_j &= L_{(2j-1)(2j)} - L_{(2j+1)(2j+2)}, \\ E_j &= \left(L_{(2j)(2j+1)} - iL_{(2j-1)(2j+1)} - iL_{(2j)(2j+2)} - L_{(2j-1)(2j+2)}\right)/2, \\ F_j &= \left(L_{(2j)(2j+1)} + iL_{(2j-1)(2j+1)} + iL_{(2j)(2j+2)} - L_{(2j-1)(2j+2)}\right)/2. \end{aligned} \tag{26a}$$

But, the bases with the subscript n are different:

$$\begin{aligned} H_n &= 2L_{(2n-1)(2n)}, \\ E_n &= L_{(2n)(2n+1)} - iL_{(2n-1)(2n+1)}, \\ F_n &= L_{(2n)(2n+1)} + iL_{(2n-1)(2n+1)}, \end{aligned} \tag{26b}$$

for SO($2n+1$), and

$$\begin{aligned} H_n &= L_{(2n-3)(2n-2)} + L_{(2n-1)(2n)}, \\ E_n &= \left(L_{(2n-2)(2n-1)} - iL_{(2n-3)(2n-1)} + iL_{(2n-2)(2n)} + L_{(2n-3)(2n)}\right)/2, \\ F_n &= \left(L_{(2n-2)(2n-1)} + iL_{(2n-3)(2n-1)} - iL_{(2n-2)(2n)} + L_{(2n-3)(2n)}\right)/2, \end{aligned} \tag{26b}$$

for SO($2n$). H_k span the Cartan subalgebra, and their eigenvalues are the components of a weight vector $\mathbf{m} = (m_1, \ldots, m_n)$:

$$H_k|\mathbf{m}\rangle = m_k|\mathbf{m}\rangle, \qquad 1 \leq k \leq n. \tag{27}$$

If the eigenstates for a given weight $\mathbf{m}$ are degeneracy, this weight is called a multiple weight, otherwise a simple one. E_k are called the raising operators and F_k the lowering ones. For an irreducible representation denoted by a Young diagram $[\mu_1, \mu_2, \ldots]$ of SO(D),

$\mu_j \geq \mu_{j+1}$, there is a highest weight $\mathbf{M} = (M_1, M_2, \ldots)$, which must be simple:

$$\begin{aligned}
&M_j = \mu_j - \mu_{j+1}, && && 1 \leq j \leq n-2,\\
&M_{n-1} = \mu_{n-1} - \mu_n, && M_n = 2\mu_n, && \text{for } \mathrm{SO}(2n+1),\\
&M_{n-1} = \mu_{n-1} - \mu_n, && M_n = \mu_{n-1} + \mu_n, && \text{for selfdual representation in } \mathrm{SO}(2n),\\
&M_{n-1} = \mu_{n-1} + \mu_n, && M_n = \mu_{n-1} - \mu_n, && \text{for antiselfdual representation in } \mathrm{SO}(2n).
\end{aligned} \tag{28}$$

We are not interested here in the spinor representations where M_n is odd for $\mathrm{SO}(2n+1)$ and $M_{n-1} + M_n$ is odd for $\mathrm{SO}(2n)$. For a given irreducible representation $[\mu_1, \mu_2, \ldots]$ of $\mathrm{SO}(D)$, we only need to consider the highest weight state $|\mathbf{M}\rangle$, which satisfies

$$H_k|\mathbf{M}\rangle = M_k|\mathbf{M}\rangle, \qquad E_k|\mathbf{M}\rangle = 0, \qquad 1 \leq k \leq n, \tag{29}$$

because its partners can be calculated by the lowering operators F_k. In this paper the highest weight state will simply be called the wave functions with the given angular momentum $[\mu, \nu]$ for simplicity.

Now, we return to our problems. Recalling the Clebsch-Gordan series in Eq. (24), we can rewrite Eq. (23) for the highest weight state $\mathbf{M}$:

$$\mathcal{Y}^{l,l',s,t}_{\mathbf{M}}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{m}} \mathcal{Y}^{l}_{\mathbf{m}}(\mathbf{x})\mathcal{Y}^{l'}_{\mathbf{M}-\mathbf{m}}(\mathbf{y})\langle l, \mathbf{m}, l', (\mathbf{M}-\mathbf{m}) \mid [(l+l'-s-2t), s], \mathbf{M}\rangle, \tag{30}$$

where the subscripts of the harmonic polynomials are changed to the weights for simplicity. $\mathcal{Y}^{l,l',s,t}_{\mathbf{M}}(\mathbf{x},\mathbf{y})$ is the highest weight state of the representation $[(l+l'-s-2t), s]$. It is a homogeneous polynomial of degrees l and l' in the components of $\mathbf{x}$ and $\mathbf{y}$, respectively. Generally speaking, some $\mathcal{Y}^{l,l',s,t}_{\mathbf{M}}(\mathbf{x},\mathbf{y})$ may be expressed as a sum where each term is a product of an internal variable ξ_j and a homogeneous polynomial $f(\mathbf{x},\mathbf{y})$ of lower degree (see p. 042108-5 in [29]). Since $\mathcal{Y}^{l,l',s,t}_{\mathbf{M}}(\mathbf{x},\mathbf{y})$ will be used as a base function for the wave function with a given angular momentum and the combinative coefficient is the function of the internal variables, in this meaning, the base function in the form of $\xi_j f(\mathbf{x},\mathbf{y})$ is not independent, and we should find out the independent and complete base functions for any given angular momentum $[\mu, \nu]$. In the following we are going to prove $\mathcal{Y}^{q,(\mu+\nu-q),\nu,0}_{\mathbf{M}}(\mathbf{x},\mathbf{y})$ and their partners, where $l = q$, $l' = \mu + \nu - q$, $s = \nu$, $t = 0$, and $\nu \leq q \leq \mu$, constitute a complete set of independent base functions for the total orbital angular momentum $[\mu, \nu]$. In other words, those total angular momentum states $\mathcal{Y}^{l,l',s,t}_{\mathbf{M}}(\mathbf{x},\mathbf{y})$ with $t > 0$ are not

independent, where the sum of the partial angular momentum quantum number l and l' is larger than $\mu+\nu$ for the total angular momentum $[\mu,\nu]$.

The highest weight for the representation $[\mu,\nu]$ is $\mathbf{M}=(\mu-\nu,\nu,0,\ldots,0)$. Removing the normalization factor in $\mathcal{Y}_{\mathbf{M}}^{q,(\mu+\nu-q),\nu,0}(\mathbf{x},\mathbf{y})$, which is irrelevant here, we can determine the explicit form for $\mathcal{Y}_{\mathbf{M}}^{q,(\mu+\nu-q),\nu,0}(\mathbf{x},\mathbf{y})$ according to its orders in the components of $\mathbf{x}$ and $\mathbf{y}$ and the property of the highest weight state (29), and denote it by the generalized harmonic polynomial $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$:

$$\begin{aligned} Q_q^{\mu\nu}(\mathbf{x},\mathbf{y}) &= \frac{X_{12}^{q-\nu}Y_{12}^{\mu-q}}{(q-\nu)!(\mu-q)!}\left(X_{12}Y_{34}-Y_{12}X_{34}\right)^{\nu} \\ &\sim \mathcal{Y}_{\mathbf{M}}^{q,(\mu+\nu-q),\nu,0}(\mathbf{x},\mathbf{y}), \qquad 0\le\nu\le q\le\mu, \\ X_{12}=x_1+ix_2, \quad X_{34}=x_3+ix_4, \quad Y_{12}=y_1+iy_2, \quad Y_{34}=y_3+iy_4. \end{aligned} \tag{31}$$

The formula for $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ holds for $D=3$ ($x_4=y_4=0$, $\nu=0$ or 1) [35,29] and $D>4$. When $D=4$ we denote the highest weight states by $Q_q^{(S)\mu\nu}(\mathbf{x},\mathbf{y})$ and $Q_q^{(A)\mu\nu}(\mathbf{x},\mathbf{y})$ for the selfdual representations and the antiselfdual representations, respectively:

$$\begin{aligned} Q_q^{(S)\mu\nu}(\mathbf{x},\mathbf{y}) &= \frac{X_{12}^{q-\nu}Y_{12}^{\mu-q}}{(q-\nu)!(\mu-q)!}\left(X_{12}Y_{34}-Y_{12}X_{34}\right)^{\nu} \\ Q_q^{(A)\mu\nu}(\mathbf{x},\mathbf{y}) &= \frac{X_{12}^{q-\nu}Y_{12}^{\mu-q}}{(q-\nu)!(\mu-q)!}\left(X_{12}Y'_{34}-Y_{12}X'_{34}\right)^{\nu} \\ X'_{34}=x_3-ix_4, &\qquad Y'_{34}=y_3-iy_4. \end{aligned} \tag{32}$$

The generalized harmonic polynomial $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ is a homogeneous polynomial of degrees q and $(\mu+\nu-q)$ in the components of $\mathbf{x}$ and $\mathbf{y}$, respectively. It is a simultaneous eigenfunction of $\bigtriangledown_{\mathbf{x}}^2$, $\bigtriangledown_{\mathbf{y}}^2$, $\bigtriangledown_{\mathbf{x}}\cdot\bigtriangledown_{\mathbf{y}}$, and the total angular momentum operator $\mathbf{L}^2$ [see Eq. (9)],

$$\begin{aligned} &\bigtriangledown_{\mathbf{x}}^2 Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})=\bigtriangledown_{\mathbf{y}}^2 Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})=\bigtriangledown_{\mathbf{x}}\cdot\bigtriangledown_{\mathbf{y}} Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})=0, \\ &\mathbf{L}^2 Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})=C_2([\mu,\nu])Q_q^{\mu\nu}(\mathbf{x},\mathbf{y}), \\ &C_2([\mu,\nu])=\mu(\mu+D-2)+\nu(\nu+D-4), \end{aligned} \tag{33}$$

where $C_2([\mu,\nu])$ is the Casimir calculated by a general formula (see (1.131) in Ref. [38]). The parity of $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ is obviously $(-1)^{\mu+\nu}$.

It is evident that $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ do not contain a function of the internal variables as a factor, neither do their partners due to the rotational symmetry. Therefore, $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ are independent base functions for the given angular momentum described by $[\mu,\nu]$. Now, we are going to prove that $(\mu-\nu+1)$ base functions $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ where $\nu\le q\le\mu$ are

complete for the angular momentum $[\mu,\nu]$. That is, $Q_q^{\mu(l-\mu)}(\mathbf{x},\mathbf{y})$ with $0 \le l-\mu \le q \le \mu$ and their partners construct a complete set of linearly independent base functions for the homogeneous polynomials of degree l in the components of $\mathbf{x}$ and $\mathbf{y}$, excluded those in the forms of $\xi_j f(\mathbf{x},\mathbf{y})$, where $f(\mathbf{x},\mathbf{y})$ is a homogeneous polynomial of degree $(l-2)$.

The number of linearly independent homogeneous polynomials of degree l in the components of $\mathbf{x}$ and $\mathbf{y}$ is

$$M_D(l) = \left(\begin{array}{c} l+2D-1 \\ 2D-1 \end{array} \right).$$

After removing those polynomials in the form $\xi_j f(\mathbf{x},\mathbf{y})$, the number $M(l)$ reduces to $K(l)$:

$$\begin{aligned} K_D(l) &= M_D(l) - 3M_D(l-2) + 3M_D(l-4) - M_D(l-6) \\ &= 4(l+D-3)\left[2l(l+2D-6) + (D-2)(2D-5)\right] \frac{(l+2D-7)!}{l!(2D-4)!}, \end{aligned} \tag{34}$$

when $l+2D \ge 7$, which only excludes one case of $l=0$ and $D=3$, where $K_3(0)=1$.

On the other hand, the number of $Q_q^{\mu(l-\mu)}(\mathbf{x},\mathbf{y})$ with $0 \le l-\mu \le q \le \mu$ and their partners can be calculated directly from Eq. (25):

$$\sum_{l/2 \le \mu \le l} (2\mu - l + 1) d_D([\mu,(l-\mu)]) = K_D(l). \tag{35}$$

Eqs. (34) and (35) are checked by Mathematica. Thus, we have proved that $(\mu-\nu+1)$ base functions $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ where $0 \le \nu \le q \le \mu$ are independent and complete for the angular momentum $[\mu,\nu]$. Any function with the angular momentum $[\mu,\nu]$ in the system can be expanded with respect to the base functions $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$, where the coefficients are functions of internal variables.

From Eq. (30), for a given total orbital angular momentum $[\mu,\nu]$ there are infinite number of wave functions $\mathcal{Y}_{\mathbf{M}}^{(q+t),(\mu+\nu+t-q),\nu,t}(\mathbf{x},\mathbf{y})$ combined from different partial angular momentum states. Now, what we have proved is that only a finite number of partial angular momentum states ($t=0$) are involved in the complete set of independent base functions $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ for a given total orbital angular momentum $[\mu,\nu]$.

V. GENERALIZED RADIAL EQUATIONS

In the preceding section we proved that any function with angular momentum $[\mu,\nu]$ in the quantum three-body system of D dimensions can be expanded with respect to the

base functions $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$

$$\Psi_{\mathbf{M}}^{[\mu,\nu]}(\mathbf{x},\mathbf{y}) = \sum_{q=\nu}^{\mu} \psi_q^{\mu\nu}(\xi_1,\xi_2,\xi_3) Q_q^{\mu\nu}(\mathbf{x},\mathbf{y}), \tag{36}$$

where the coefficients $\psi_q^{\mu\nu}(\xi_1,\xi_2,\xi_3)$ are called the generalized radial functions. When substituting Eq. (36) into the Schrödinger equation (22), the main calculation in the derivation is to apply the Laplace operator (21) to the function $\Psi_{\mathbf{M}}^{[\mu,\nu]}(\mathbf{x},\mathbf{y})$. The calculation consists of three parts. The first is to apply the Laplace operator to the generalized radial functions $\psi_q^{\mu\nu}(\xi_1,\xi_2,\xi_3)$, which can be calculated by replacement of variables:

$$\begin{aligned}\bigtriangledown^2 \psi_q^{\mu\nu}(\xi_1,\xi_2,\xi_3) = \Big\{ & 4\xi_1\partial_{\xi_1}^2 + 4\xi_2\partial_{\xi_2}^2 + 2D\left(\partial_{\xi_1}+\partial_{\xi_2}\right) + \left(\xi_1+\xi_2\right)\partial_{\xi_3}^2 \\ & + 4\xi_3\left(\partial_{\xi_1}+\partial_{\xi_2}\right)\partial_{\xi_3}\Big\}\psi_q^{\mu\nu}(\xi_1,\xi_2,\xi_3),\end{aligned} \tag{37}$$

where ∂_ξ denotes $\partial/\partial\xi$ and so on. The second is to apply it to the generalized harmonic polynomials $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$. This part is vanishing because $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ satisfies the Laplace equation. The last is the mixed application

$$2\left\{\left(\partial_{\xi_1}\psi_q^{\mu\nu}\right)2\mathbf{x} + \left(\partial_{\xi_3}\psi_q^{\mu\nu}\right)\mathbf{y}\right\}\cdot\bigtriangledown_{\mathbf{x}} Q_q^{\mu\nu} + 2\left\{\left(\partial_{\xi_2}\psi_q^{\mu\nu}\right)2\mathbf{y} + \left(\partial_{\xi_3}\psi_q^{\mu\nu}\right)\mathbf{x}\right\}\cdot\bigtriangledown_{\mathbf{y}} Q_q^{\mu\nu}. \tag{38}$$

From the definition (31) for $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ we have

$$\begin{aligned} \mathbf{x}\cdot\bigtriangledown_{\mathbf{x}} Q_q^{\mu\nu} &= qQ_q^{\mu\nu}, & \mathbf{y}\cdot\bigtriangledown_{\mathbf{y}} Q_q^{\mu\nu} &= (\mu+\nu-q)Q_q^{\mu\nu} \\ \mathbf{y}\cdot\bigtriangledown_{\mathbf{x}} Q_q^{\mu\nu} &= (\mu-q+1)Q_{q-1}^{\mu\nu}, & \mathbf{x}\cdot\bigtriangledown_{\mathbf{y}} Q_q^{\mu\nu} &= (q-\nu+1)Q_{q+1}^{\mu\nu}. \end{aligned} \tag{39}$$

Hence, we obtain the generalized radial equation, satisfied by the $(\mu-\nu+1)$ functions $\psi_q^{\mu\nu}(\xi_1,\xi_2,\xi_3)$:

$$\begin{aligned}\bigtriangledown^2\psi_q^{\mu\nu} + 4q\partial_{\xi_1}\psi_q^{\mu\nu} + 4(\mu+\nu-q)\partial_{\xi_2}\psi_q^{\mu\nu} + 2(\mu-q)\partial_{\xi_3}\psi_{q+1}^{\mu\nu} + 2(q-\nu)\partial_{\xi_3}\psi_{q-1}^{\mu\nu} \\ = -2\left(E-V\right)\psi_q^{\mu\nu},\end{aligned} \tag{40}$$

where $\bigtriangledown^2\psi_q^{\mu\nu}$ is given in Eq. (37). Only three invariant variables ξ_1, ξ_2 and ξ_3 are involved both in the equations and in the functions. When $D=4$, Eq. (40) holds for the generalized radial functions either in $[(S)\mu,\nu]$ or in $[(A)\mu,\nu]$, because two representations incorporate to one irreducible representation of the O(4) group when the space inversion is considered. When $D=3$, the equations for the functions in $[\mu,0]$ and in $[\mu,1]$ are different although two representations $[\mu,0]$ and $[\mu,1]$ are equivalent, because the functions have different parity.

At last, we discuss rotational variables and the volume element of the configuration space. We fix the body-fixed frame such that $\mathbf{x}$ is parallel with its Dth axis, and $\mathbf{y}$ is located in its $(D-1)D$ hyperplane with a non-negative $(D-1)$th component. That is, in the body-fixed frame, the nonvanishing components of two Jacobi coordinate vectors $\mathbf{x}'$ and $\mathbf{y}'$ are

$$x'_D = \xi_1^{1/2}, \quad y'_{D-1} = \left(\xi_2 - \xi_3^2/\xi_1\right)^{1/2}, \quad y'_D = \xi_3 \xi_1^{-1/2}. \tag{41}$$

Let $R = R^{(1)}R^{(2)} \in \mathrm{SO}(D)$ rotate the center-of-mass frame to the body-fixed frame:

$$\begin{aligned} R^{(1)} &= R_{12}(\theta_1)R_{31}(\theta_2)R_{43}(\theta_3)R_{54}(\theta_4)\ldots R_{D(D-1)}(\theta_{D-1}), \\ R^{(2)} &= R_{12}(\varphi_1)R_{31}(\varphi_2)R_{43}(\varphi_3)R_{54}(\varphi_4)\ldots R_{(D-1)(D-2)}(\varphi_{D-2}), \\ R\mathbf{x}' &= \mathbf{x}, \qquad R\mathbf{y}' = \mathbf{y}, \end{aligned} \tag{42}$$

where, for example, $R_{12}(\theta)$ is a rotation on the hyperplane with the first and the second axes through θ angle:

$$R_{12}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \mathbf{1}_{D-2} \end{pmatrix}.$$

$(D-1)$ θ_j and $(D-2)$ φ_k are the rotational variables, called the generalized Euler angles. Through a straightforward calculation, we obtain

$$\begin{aligned} x_1 + ix_2 &= \xi_1^{1/2} e^{i\theta_1} \prod_{a=2}^{D-1} \sin\theta_a, \qquad x_3 + ix_4 = \xi_1^{1/2} \left(\cos\theta_2 \sin\theta_3 + i\cos\theta_3\right) \prod_{a=4}^{D-1} \sin\theta_a, \\ y_1 + iy_2 &= \xi_3 \xi_1^{-1/2} e^{i\theta_1} \prod_{a=2}^{D-1} \sin\theta_a + \left(\xi_2 - \xi_3^2/\xi_1\right)^{1/2} e^{i\theta_1} \\ &\quad \times \left\{ i \prod_{a=1}^{D-2} \sin\varphi_a + \sum_{a=1}^{D-2} \cos\theta_{a+1} \cos\varphi_a \left(\prod_{b=2}^{a} \sin\theta_b \right) \left(\prod_{c=a+1}^{D-2} \sin\varphi_c \right) \right\}, \\ y_3 + iy_4 &= \xi_3 \xi_1^{-1/2} \left(\cos\theta_2 \sin\theta_3 + i\cos\theta_3\right) \prod_{a=4}^{D-1} \sin\theta_a + \left(\xi_2 - \xi_3^2/\xi_1\right)^{1/2} \\ &\quad \times \left\{ -\cos\varphi_1 \sin\theta_2 \prod_{a=2}^{D-2} \sin\varphi_a + \left(\cos\theta_2 \cos\theta_3 - i\sin\theta_3\right) \cos\varphi_2 \prod_{a=3}^{D-2} \sin\varphi_a \right. \\ &\quad \left. + \left(\cos\theta_2 \sin\theta_3 + i\cos\theta_3\right) \sum_{a=3}^{D-2} \cos\theta_{a+1} \cos\varphi_a \left(\prod_{b=4}^{a} \sin\theta_b \right) \left(\prod_{c=a+1}^{D-2} \sin\varphi_c \right) \right\}, \end{aligned} \tag{43}$$

where $\prod_{a=b+1}^{b} F_a = 1$. The volume element of the configuration space is

$$\prod_{j=1}^{D} dx_j dy_j = \frac{1}{4} \left(\xi_1\xi_2 - \xi_3^2\right)^{(D-3)/2} d\xi_1 d\xi_2 d\xi_3 \prod_{j=1}^{D-1} \left(\sin\theta_j\right)^{j-1} d\theta_j \prod_{k=1}^{D-2} \left(\sin\varphi_k\right)^{k-1} d\varphi_k. \tag{44}$$

VI. CONCLUSIONS

After separating the motion of center of mass, we have defined the homogeneous polynomial $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ of degree q and $(\mu+\nu-q)$ in the components of the Jacobi coordinate vectors $\mathbf{x}$ and $\mathbf{y}$, respectively. $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ is a solution of the Laplace equation. We have proved that $(\mu-\nu+1)$ generalized harmonic polynomials $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$ constitute a complete set of independent base functions for the total orbital angular momentum $[\mu,\nu]$. Any wave function with the given angular momentum in the system can be expanded with respect to the base functions, where the coefficients are the functions of the internal variables, called the generalized radial functions. The three-body Schrödinger equation with a spherically symmetric potential V in D dimensions reduces to the generalized radial equations satisfied by the generalized radial functions. Only three internal variables are involved in the functions and equations. The number of both the functions and the equations for the given angular momentum $[\mu,\nu]$ is finite and equal to $(\mu-\nu+1)$. Only a finite number of partial angular momentum states are involved in constructing the generalized harmonic polynomials $Q_q^{\mu\nu}(\mathbf{x},\mathbf{y})$, and the contributions from the remaining partial angular momentum states have been incorporated into those from the generalized radial functions.

The generalization of this method to a quantum N-body system in D-dimensions is straightforward. The difficulty is how to obtain the unified forms for the generalized harmonic polynomials, because it needs $D-1$ vectors to determine the body-fixed frame and there are $N-1$ Jacobi coordinate vectors. The cases with $N<D$ are very different to the cases with $N\geq D$. We will study this problem elsewhere.

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